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The Modification of Exponentiated Exponential Distribution Using Odd Exponentiated Half Logistic-G Technique: Properities and Application

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Abstract
This paper introduces a new life time distribution called Odd Exponentiated Half Logistic Exponentiated Exponential (OEHL-EE) using Odd Exponentiated Half Logistic –G generator of statistical distributions. Some mathematical properties of the proposed OEHL-EE distribution are derived such as moments, coefficient of variation, skewness, kurtosis, moment generating function, quantile function, Renyi, entropy, and order statistics. The estimation of the OEHL-EE parameters is conducted using maximum likelihood estimation method. We compared the fits of the OEHL-EE distribution with some selected existing models and the results show that the proposed OEHL-EE model best fitted to both simulated and real-life data than other considered existing models when compare under different model

Key Words: Zero- Odd Exponentiated Half Logistic Exponentiated Exponential ; Monents; Generating Function, Maximum Likelihood; Quantile Function; Renyi; Entropy.

I. Introduction

Addition of new shape parameter(s) to an existing probability model expands the model into a larger family of distributions and this provides a significantly skewed and heavy-tails probability distribution models. The extended or generalized model can provide better flexibility in modelling lifetime data for better actualization of phenomenon contained in the dataset. An approach to the construction of flexible parametric models is to insert appropriate competing models into a larger model by adding shape parameter(s) (Aryal &Tokos, 2009).

Statistical distributions are frequently used to describe real-life events. This is due to its efficiency, the theory of statistical distribution is extensively studied, and new distributions are also created. There is a strong desire among Statisticians to create statistical distributions that are more flexible and reliable in nature.

Many different types of generalized distributions have been developed and applied to various phenomena. In many fields of study, such as economics, engineering, biological studies and environmental sciences, several continuous univariate distributions have been extensively used for data modelling. However, applied areas such as finance, lifetime analysis and insurance clearly require extended forms of these distributions. So, various classes of distributions have been developed by extending common families of continuous distributions.

A new improvement for creating and extending traditional distributions is the generated families of continuous distributions. The newly generated families have been extensively researched in a variety of fields, and they provided greater application flexibility. In many practical situations, classical distributions do not provide adequate fits to real data. Therefore, there has been an increased interest in developing more flexible and robust distributions through extending the classical distributions, introducing additional shape parameters to the baseline model. The techniques for modifying the classical distributions are usually referred to as generators in literature and are capable for improving the goodness of fit of the modified distribution. These features have been established by the results of many generators (Cordeiro and de Castro, 2011; Jones, 2009; Eugene *et al.,* 2002).Alzaatreh*et al*. (2013) proposed an extension of the beta-generated family of distributions developed by Eugene *et al*. (2002) and called it, the transformed –transformer (T-X) family of distributions. According to Alzaatreh*et al.* (2013), the Cumulative distribution function (CDF) of the T-X family is defined as

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$$G(x)=\int_0^{-\log(1-F(x))} r(t)dt = R(t) = R\{1-\log(1-F(x))\ \},$$

where r(t) is the probability density function (PDF) of the random variable T, R(t) is the CDF of T and F(x) is the CDF of the random variable X. Although, the T-X method have been embraced by several researchers. Alzaghal *et al*. (2013) proposed an The CDF of this new family is given by

extension of it in order to improve on some of the drawbacks of the T-X method. The new family defined by Alzagahal *et al*. (2013) is called the exponentiated T-X family of the distributions.

$$G(x)=\int_0^{-\log\left(1-F^c(x)\right)} r(t)dt = R(t) = R\left\{1-\log\left(1-F^c(x)\right)\right\}$$

where c is a shape parameter introduced in the family to make it more flexible. However, both the T-X family and the exponentiated T-X family still have some drawbacks that need to be addressed.

The OEHL-G is a member of the family of logistic distributions which was introduced by Afify et al. (2017). It is a flexible family of distributions that oversimplifies the baseline distribution through the use of odd ratio transformation and exponentiation.

Definition: Suppose that T is a lifetime random variable with a certain continuous cumulative distribution function (cdf), $G(x;\xi)$. The odds ratio that an object which follows the lifetime T will die at time x is $\frac{G(x;\xi)}{1-G(x;\xi)}$. If the variability of these odds of death is represented by the random variable X and assumed it follows OEHL-G distributions then according to Afify et al. (2017), the cdf of OEHL-G is defined as

$$F(x;\alpha,\lambda,\xi) = \left[\frac{1-exp\left(\frac{-\lambda(G(x;\xi))}{1-G(x;\xi)}\right)}{1+exp\left(\frac{-\lambda(G(x;\xi))}{1-G(x;\xi)}\right)}\right]^{\alpha} \quad (1)$$

and the corresponding probability density function (pdf) of the OEHL-G family of distributions is obtained as

$$f(x;\alpha,\lambda,\xi) = \frac{2\alpha\lambda(g(x;\xi))\left[exp\left(\frac{-\lambda(G(x;\xi))}{1-G(x;\xi)}\right)\right]\left[1-exp\left(\frac{-\lambda(G(x;\xi))}{1-G(x;\xi)}\right)\right]^{\alpha-1}}{[1-G(x;\xi)]^2\left[1+exp\left(\frac{-\lambda(G(x;\xi))}{1-G(x;\xi)}\right)\right]^{\alpha+1}} \quad (2)$$

where,

$G(x;\xi)$ is the cdf of baseline distribution which depends on a parameter vector ξ,

$g(x;\xi)$ is the pdf of baseline distribution which depends on a parameter vector ξ,

α is a shape parameter; $\alpha > 0$,

λ is a shape parameter; $\lambda > 0$,

$x > 0$.

The aim of this study is to modify exponentiated exponential distribution using odd half logistic-G Technique in order to make more efficient in fitting right skewed data and increases its flexibility.

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II. Odd Exponentiated Half Logistic Exponentiated Exponential (OEHLEE)

Exponentiated Exponential (EE) Distribution

Definition: A random variable X is said to follow exponentiated exponential (EE)

distribution with parameters β and γ, if its cdf has the following form;

$$G(x) = \left(1 - exp(-\beta x)\right)^{\gamma} \tag{3}$$

The corresponding pdf of the EE distribution is given as

$$g(x) = \gamma\beta\left(exp(-\beta x)\right)\left(1 - exp(-\beta x)\right)^{\gamma-1} \quad x > 0;\ \beta, \gamma > 0. \tag{4}$$

The EE distribution in equation (3) has shape parameters γ, and scale parameter β. Note that the shape parameter γ represent the exponential family when it is equaled to 1 (Gupta and Kundu, 2001).

Suppose a random variable X follows an OEHL-EE distribution with parameters $\alpha, \lambda, \gamma, \beta$ and using the cdf and pdf of EE distribution as the baseline distribution then the cdf of OEHL-EE distribution is obtained by substituting equation (3) in equation (1); and it is defined as

$$F(x; \alpha, \lambda, \gamma, \beta) = \left[\frac{1 - exp\left(\frac{-\lambda\left(1 - exp(-\beta x)\right)^{\gamma}}{1 - \left(1 - exp(-\beta x)\right)^{\gamma}}\right)}{1 + exp\left(\frac{-\lambda\left(1 - exp(-\beta x)\right)^{\gamma}}{1 - \left(1 - exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha} \tag{5}$$

The corresponding pdf of OEHL-EE distribution is obtained by differentiating (5) with respect to x or by substituting equations (3) and (4) in equation (2), we have

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda\left(\gamma\beta\left(exp(-\beta x)\right)\left(1 - exp(-\beta x)\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1 - exp(-\beta x)\right)^{\gamma}}{1 - \left(1 - exp(-\beta x)\right)^{\gamma}}\right)\right]\left[1 - exp\left(\frac{-\lambda\left(1 - exp(-\beta x\right.\right.\right.}{\left[1 - \left(1 - exp(-\beta x)\right)^{\gamma}\right]^{2}\left[1 + exp\left(\frac{-\lambda\left(1 - exp(-\beta x)\right)^{\gamma}}{1 - \left(1 - exp(-\beta x)\right)^{\gamma}}\right)\right]^{\alpha+1}} \tag{6}$$

$$x \in (0, \infty);\ \alpha, \lambda, \gamma, \beta > 0$$

where, α, λ, and γ are the shape parameters; β is the scale parameter.

Hazard Function

The hazard function is also known as the failure or risk function and is the probability that a system or an individual or a unit will fail or die for an interval of time.

The hazard function is defined as

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{f(x)}{S(x)} \tag{7}$$

where, $F(x)$ and $f(x)$ are the cdf and pdf of a baseline (parent) distribution, respectively.

The hazard function of OEHL-EE distribution is given as

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$$h(x) = \frac{2\alpha\lambda\left(\gamma\beta\left(exp(-\beta x)\right)\left(1-exp(-\beta x)\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda\left(1-exp(-\beta\right.\right.\right.}{1-\left(1-exp(-\right.\right.\right.}}{\left\{1-\left\{\left[\frac{1-exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}\right\}\right\}\left\{\left[1-\left(1-exp(-\beta x)\right)^{\gamma}\right]^{2}\left[1+exp\left(\frac{-\lambda\left(1-exp(-\beta\right.\right.\right.}{1-\left(1-exp(-\right.\right.\right.}} \quad (8)$$

Reversed Hazard Function of OEHL-EE Distribution

The reversed hazard function of OEHL-EE distribution is given as

$$r(x) = \frac{2\alpha\lambda\left(\gamma\beta\left(exp(-\beta x)\right)\left(1-exp(-\beta x)\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda\left(1-exp(-\beta\right.\right.}{1-\left(1-exp(-\beta\right.\right.}}{\left\{\left[\frac{1-exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}\right\}\left\{\left[1-\left(1-exp(-\beta x)\right)^{\gamma}\right]^{2}\left[1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right.\right.\right.} \quad (9)$$

Cumulative Hazard Function of OEHL-EE Distribution

The cumulative hazard function of OEHL-EE distribution is given as

$$H(x) = ln\left[1-\left\{\left[\frac{1-exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}\right\}\right] \quad (10)$$

Survival or Reliability Function of OEHL-EE Distribution

The survival function of OEHL-EE distribution is given as

$$S(x) = 1-\left\{\left[\frac{1-exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}\right\} \quad (11)$$

Odd Function of OEHL-EE Distribution

The odd function of OEHL-EE distribution is given as

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$$ODD(x) = \frac{\left[\dfrac{1-exp\left(\dfrac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\dfrac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}}{1-\left\{\left[\dfrac{1-exp\left(\dfrac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}{1+exp\left(\dfrac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)}\right]^{\alpha}\right\}} \quad (12)$$

Plots of the density function (9) can be represented through Figure 1. As it seems from Figure 1that the pdf of **OEHL-EE** can take different shapes according to different values of α and λ. It can be symmetric, right skewed, unimodal

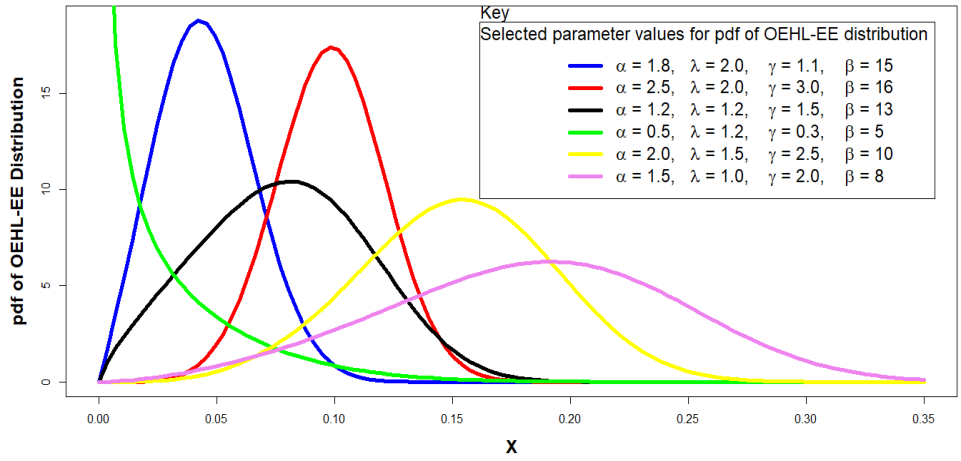


Figure 1: Plot of pdf for some selected parameter values of OEHL-EE distribution

Plots of cumulative distribution function (8) for the OEHL-EE distribution are displayed in Figure 2 The cdf graph tend to move at a constant value from 0 to 1 on the x-axis before projecting to one on the y-axis

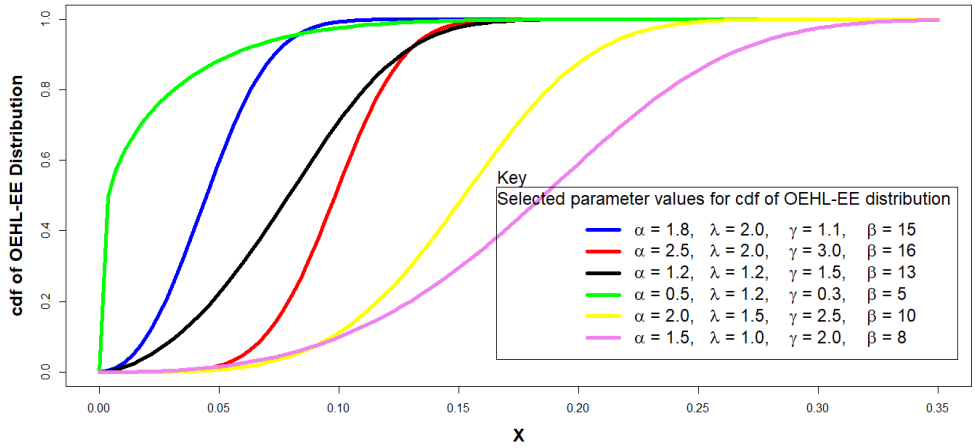


Figure 2: Plot of cdf for some selected parameter values of OEHL-EE distribution

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Plots of hazard function (10) for the **OEHL-EE** distribution are displayed in Figure 3. It was examine from Figure 3 show that the shape of the h(x) of the **OEHL-EE** shows the hazard functions can be constant, increasing or decreases, depending on the shape and scale parameters.

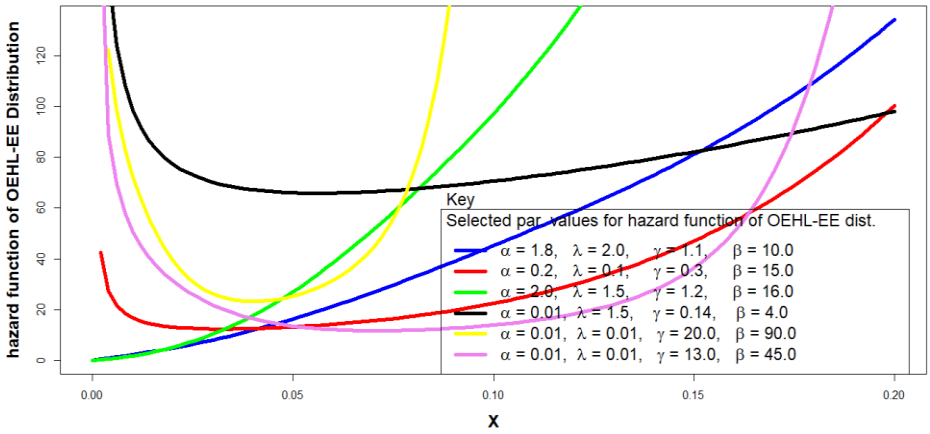


Figure 3: Plot of hazard function for some selected parameter values of OEHL-EE distribution

Plots of survival function (13) for the **OEHL-EE** distribution are displayed in Figure 4. It was deduced from Figure 4 that the shape parameters has strong effect on the survival plot. A decrease in the shape parameter tend to reduce the movement of the line on the y-axis and no plot is exceeding the bench mark of 1:

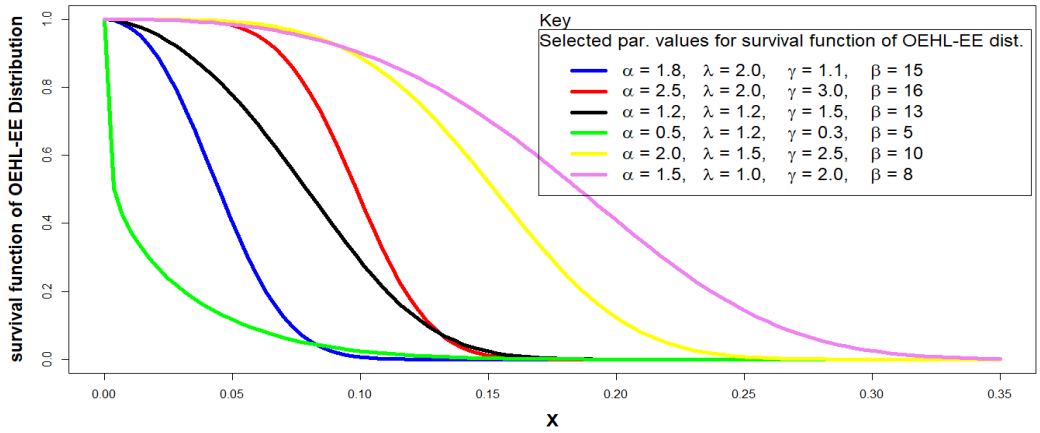


Figure 4: Plot of survival function for some selected parameter values of OEHL-EE distribution

III. Mathematical Properties

This section provide some mathematical properties of the OEHL-EE

Moment about origin for the proposed OEHL-EE Distribution

The k^{th} moments of OEHL-EE distribution is given as

$$\mu'_k = E(X^k) = \frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}} \tag{13}$$

The *kth* Moments for $k = 1,\ 2,\ 3,\ \text{and}\ 4$

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Using equations (13), the corresponding moments for $k = 1,\ 2,\ 3,\ \text{and}\ 4$ are as follows

for $\mathbf{k = 1}$

$$E(X) = \frac{2\alpha\Gamma\left(\frac{1}{\gamma} + \alpha\right)}{\beta(2^{(\alpha+1)})\lambda^{\left(\frac{1}{\gamma}\right)}} \tag{14}$$

for $\mathbf{k = 2}$

$$E(X^2) = \frac{2\alpha\Gamma\left(\frac{2}{\gamma} + \alpha\right)}{\beta^2(2^{(\alpha+1)})\lambda^{\left(\frac{2}{\gamma}\right)}} \tag{15}$$

for $\mathbf{k = 3}$

$$E(X^3) = \frac{2\alpha\Gamma\left(\frac{3}{\gamma} + \alpha\right)}{\beta^3(2^{(\alpha+1)})\lambda^{\left(\frac{3}{\gamma}\right)}} \tag{16}$$

for $\mathbf{k = 4}$

$$E(X^4) = \frac{2\alpha\Gamma\left(\frac{4}{\gamma} + \alpha\right)}{\beta^4(2^{(\alpha+1)})\lambda^{\left(\frac{4}{\gamma}\right)}} \tag{17}$$

Mean and Variance of OEHL-EE Distribution

Mean of OEHL-EE

The mean of a given random variable X is given as

$$E(X) = \mu = \mu_1' \tag{18}$$

Substituting equation (14) in equation (18), we have the mean of OEHL-EE distribution; that is, the first moment about origin for OEHL-EE distribution is the mean of OEHL-EE. It is defined as

$$\mu_1^{'} = E(X) = \frac{2\alpha\Gamma\left(\frac{1}{\gamma} + \alpha\right)}{\beta(2^{(\alpha+1)})\lambda^{\left(\frac{1}{\gamma}\right)}} \tag{19}$$

Variance of OEHL-EE

The variance of a given random variable X is given as

$$Var(X) = \mu_2' - (\mu_1')^2 \tag{20}$$

Substituting equations (14) and (15) in equation (20), we have the variance of OEHL-EE distribution. That is, the second moment of OEHL-EE minus the first moment of it; and it is defined as

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$$Var(X) = \frac{2\alpha\Gamma\left(\frac{2}{\gamma}+\alpha\right)}{\beta^2(2^{(\alpha+1)})\lambda^{\left(\frac{2}{\gamma}\right)}} - \left\{\frac{2\alpha\Gamma\left(\frac{1}{\gamma}+\alpha\right)}{\beta(2^{(\alpha+1)})\lambda^{\left(\frac{1}{\gamma}\right)}}\right\}^2 \quad (21)$$

The j^{th} Moment about the Mean for OEHL-EE Distribution

The j^{th} moment about the mean for OEHL-EE distribution is given as

$$E(x-\mu)^j = \sum_{k=0}^{j}\binom{j}{k}(-u)^{j-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right) \quad (22)$$

The j^{th} Moment about the Mean for $j = 1,\ 2,\ 3,\ \text{and}\ 4$

for $j = 1$

$$E(x-\mu) = \sum_{k=0}^{1}\binom{1}{k}(-u)^{1-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right) \quad (23)$$

for $j = 2$

$$E(x-\mu)^2 = \sum_{k=0}^{2}\binom{2}{k}(-u)^{2-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right) \quad (24)$$

for $j = 3$

$$E(x-\mu)^3 = \sum_{k=0}^{3}\binom{3}{k}(-u)^{3-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right) \quad (25)$$

for $j = 4$

$$E(x-\mu)^4 = \sum_{k=0}^{4}\binom{4}{k}(-u)^{4-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^k(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right) \quad (26)$$

Coefficient of Variation (CV)

The coefficient of variation (CV) is defined as

$$CV = \frac{\sqrt{\sigma^2}}{\mu} \quad (21)$$

Recall that the second moment about the mean is equal to variance, σ^2 and the first moment about the origin is equal to mean, μ; then using equations (8) and (18) the coefficient of variation of OEHL-EE distribution is given as

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$$CV = \frac{\left(\beta\left(2^{(\alpha+1)}\right)\lambda^{\left(\frac{1}{\gamma}\right)}\right)\sqrt{\sum_{k=0}^{2}\binom{2}{k}(-u)^{2-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}\left(2^{(\alpha+1)}\right)\lambda^{\left(\frac{k}{\gamma}\right)}}\right)}}{2\alpha\Gamma\left(\frac{1}{\gamma}+\alpha\right)} \quad (22)$$

Moment Generating Function (MGF) of OEHL-EE Distribution

Moment Generating Function (MGF) is a way of organizing all the moments into one mathematical quantity. The moment-generating function of a random variable X is given as

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx \quad (29)$$

Let X be a random variable with the OEHL-EE distribution with parameters α, λ, γ and β

$$\int_0^{\infty} f(x;\alpha,\lambda,\gamma,\beta\ )dx$$

$$= \int_0^{\infty} \frac{2\alpha\lambda\left(\gamma\beta(exp(-\beta x))(1-exp(-\beta x))^{\gamma-1}\right)\left[exp\left(\frac{-\lambda(1-exp(-\beta x))^{\gamma}}{1-(1-exp(-\beta x))^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda(1-exp(-\beta x))^{\gamma}}{1-(1-exp(-\beta x))^{\gamma}}\right)\right]^{\alpha}}{\left[1-(1-exp(-\beta x))^{\gamma}\right]^{2}\left[1+exp\left(\frac{-\lambda(1-exp(-\beta x))^{\gamma}}{1-(1-exp(-\beta x))^{\gamma}}\right)\right]^{\alpha+1}}$$

$$x \in (0,\infty);\ \alpha,\lambda,\gamma, \text{and } \beta\ > 0$$

The moment generating function (MGF) of the OEHL-EE Distribution is given as

$$M_x(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}\left(2^{(\alpha+1)}\right)\lambda^{\left(\frac{k}{\gamma}\right)}} \quad (30)$$

Rényi Entropy of the OEHL-EE Distribution

Assume X is a random variable that follows OEHL-EE distribution with parameters α, λ, γ and β; that is, $X \sim OEHL - EE(\alpha, \lambda, \gamma, \beta)$.

The Rényi Entropy of a random variable X that follows OEHL-EE distribution is given by

$$I_\theta(X) = \frac{1}{1-\theta}log\left\{\frac{(2\alpha\lambda\gamma\beta)^{\theta}}{\beta}\sum_{j=0}^{\infty}\frac{\theta^s}{s!}(-\lambda)^k\binom{\theta(\alpha-1)}{n}\binom{-\theta(\alpha+1)}{m}\binom{k+2\theta+l-1}{l}\frac{\Gamma(\theta)\Gamma(\theta(\gamma-1)+\gamma j+1)}{\Gamma(\theta+\theta(\gamma-1)+\gamma j+1)}\right\} \quad (31)$$

Order Statistics

Order statistics refer to a sample of random variables arranged in ascending order, from the smallest to the largest value. It plays an important role in various statistical applications, including robust statistical estimation, outlier detection, probability distribution characterization, goodness-of-fit tests, entropy estimation, censored samples analysis, reliability assessment, quality control, and material strength evaluation. Consider a random

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sample $X_1, X_2, \cdots, X_n$ drawn from a distribution with probability density function (pdf), $f(x)$. The corresponding order statistics, denoted as $X_{1:n}, X_{2:n}, \cdots, X_{i:n}$, represent the sorted values of the sample, where $X_{(1:n)}$ is the smallest observation and $X_{(n:n)}$ is the largest.

The pdf, $f_{i:n}(x)$ of the i^{th} order statistic can be defined as

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x) F(x)^{i-1} [1 - F(x)]^{n-i} \tag{32}$$

Order Statistic of OEHL-EE Distribution

The density function of the kth order statistics for a given random samples $x_1, x_2, \cdots, x_n$ from the cdf and pdf of OEHL-EE distribution is given as

$$f_{k:n}(x) = \frac{\mathrm{n!}}{(k-1)!\,(\mathrm{n}-k)!} \frac{2\alpha\lambda\left(\gamma\beta\left(e^{-\beta x}\right)\left(1-e^{-\beta x}\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1-e^{-\beta x}\right)^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda\left(1-e^{-\beta x}\right)^{\gamma}}{1-(1-e^{-\beta x})}\right.\right.}{[1-(1-e^{-\beta x})^{\gamma}]^2\left[1+exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)\right]^{\alpha+1}}$$
$$\times\left[\left(\frac{1-exp\left(\frac{-\lambda\left(1-e^{-\beta x}\right)^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}{1+exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}\right)^{\alpha}\right]^{k-1}\left[1-\left(\frac{1-exp\left(\frac{-\lambda\left(1-e^{-\beta x}\right)^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}{1+exp\left(\frac{-\lambda(1-e^{-\beta x})^{\gamma}}{1-(1-e^{-\beta x})^{\gamma}}\right)}\right)^{\alpha}\right]^{n-k} \tag{33}$$

Two approaches are used in obtaining the Skewness and kurtosis of the OEHL-EE distribution. These approaches include the measure of kurtosis (k.u) and skewness (s.k) based on moments and quantiles. In the moments based approach,

$$\mathrm{CS} = \frac{\sum_{k=0}^{3}\binom{3}{k}(-u)^{3-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right)}{\left(\sum_{k=0}^{2}\binom{2}{k}(-u)^{2-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right)\right)^{\frac{3}{2}}} \tag{34}$$

$$CK = \frac{\sum_{k=0}^{3}\binom{4}{k}(-u)^{4-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right)}{\left(\sum_{k=0}^{3}\binom{2}{k}(-u)^{2-k}\left(\frac{2\alpha\Gamma\left(\frac{k}{\gamma}+\alpha\right)}{\beta^{k}(2^{(\alpha+1)})\lambda^{\left(\frac{k}{\gamma}\right)}}\right)\right)^{2}} \tag{35}$$

IV. Parameter Estimation

From parameter estimation obtain the maximum likelihood estimator for OEHL-EE in the case of complete sample and simulation study is performed.

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Parameters Estimation of the OEHL-EE Distribution using Maximum Likelihood Estimation (MLE) Method

Let $x_1, x_2, \cdots, x_n$ be the random sample of size n from the OEHL-EE distribution with unknown parameters α, λ, γ and β.

Let $\Theta = (\alpha, \lambda, \gamma, \beta)^T$ be the p x 1 parameter vector.

The pdf of OEHL-EE distribution is given as

$$f(x; \alpha, \lambda, \gamma, \beta) = \frac{2\alpha\lambda\left(\gamma\beta\left(exp(-\beta x)\right)\left(1-exp(-\beta x)\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)\right]^{\alpha-1}}{\left[1-\left(1-exp(-\beta x)\right)^{\gamma}\right]^2\left[1+exp\left(\frac{-\lambda\left(1-exp(-\beta x)\right)^{\gamma}}{1-\left(1-exp(-\beta x)\right)^{\gamma}}\right)\right]^{\alpha+1}} \quad (36)$$

$$x \in (0, \infty); \alpha, \lambda, \gamma, \beta > 0$$

where, α, λ, and γ are the shape parameters; β is the scale parameter.

For determining the MLE of Θ using (36)

$$L(\Theta) = \prod_{i=1}^{n} f(x_i; \alpha, \lambda, \gamma, \beta)$$

$$= \prod_{i=1}^{n}\left(\frac{2\alpha\lambda\left(\gamma\beta e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)^{\gamma-1}\right)\left[exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]\left[1-exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha-1}}{[1-(1-e^{-\beta x_i})^{\gamma}]^2\left[1+exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha+1}}\right)$$

$$= \frac{2^n\alpha^n\lambda^n\gamma^n\beta^n\prod_{i=1}^{n}e^{-\beta x_i}\prod_{i=1}^{n}\left(1-e^{-\beta x_i}\right)^{\gamma-1}\prod_{i=1}^{n}\left[exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]\prod_{i=1}^{n}\left[1-exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha-1}}{\prod_{i=1}^{n}[1-(1-e^{-\beta x_i})^{\gamma}]^2\prod_{i=1}^{n}\left[1+exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha+1}} \quad (37)$$

Taking the log-likelihood of (3.163), i.e., $\log L(\Theta)$, we have

$$\log L(\Theta) = \log\left(\frac{2^n\alpha^n\lambda^n\gamma^n\beta^n\prod_{i=1}^{n}e^{-\beta x_i}\prod_{i=1}^{n}\left(1-e^{-\beta x_i}\right)^{\gamma-1}\prod_{i=1}^{n}\left[exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]\prod_{i=1}^{n}\left[1-exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha-1}}{\prod_{i=1}^{n}[1-(1-e^{-\beta x_i})^{\gamma}]^2\prod_{i=1}^{n}\left[1+exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right]^{\alpha+1}}\right) \quad (38)$$

$$= n\log 2 + n\log\alpha + n\log\lambda + n\log\gamma + n\log\beta - \beta\sum_{i=1}^{n}x_i + (\gamma-1)\sum_{i=1}^{n}\log\left(1-e^{-\beta x_i}\right) + \sum_{i=1}^{n}\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right) - 2\sum_{i=1}^{n}\log\left(1-\left(1-e^{-\beta x_i}\right)^{\gamma}\right) - (\alpha+1)\sum_{i=1}^{n}\log\left(1+exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right) \quad (39)$$

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Differentiating (39) with respect to each parameter $\alpha, \lambda, \gamma,$ and β, and equating each of the derivative to zero

$$\frac{d\log L(\Theta)}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1 - exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right) - \sum_{i=1}^{n} \log\left(1 + exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)\right) = 0 \tag{40}$$

$$\frac{d\log L(\Theta)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{\left(1-e^{-\beta x_i}\right)^{\gamma}}{(1-(1-e^{-\beta x_i})^{\gamma})^2} \left\{ exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) - \frac{(\alpha-1)exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)}{1-exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)} - \frac{(\alpha+1)}{1+exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)} \right\} = 0 \tag{41}$$

$$\frac{d\log L(\Theta)}{d\gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \log\left(1-e^{-\beta x_i}\right) - 2\sum_{i=1}^{n} \frac{\left(1-e^{-\beta x_i}\right)^{\gamma} \log\left(1-e^{-\beta x_i}\right)}{1-(1-e^{-\beta x_i})^{\gamma}} - \sum_{i=1}^{n} \frac{\lambda\left(1-e^{-\beta x_i}\right)^{\gamma} \log\left(1-e^{-\beta x_i}\right)}{1-(1-e^{-\beta x_i})^{\gamma^2}} \left\{ exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right) - \frac{(\alpha-1)exp\left(\frac{-\lambda\left(1-e^{-\beta x_i}\right)^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)}{1-exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)} - \frac{(\alpha+1)}{1+exp\left(\frac{-\lambda(1-e^{-\beta x_i})^{\gamma}}{1-(1-e^{-\beta x_i})^{\gamma}}\right)} \right\} = 0 \tag{42}$$

$$\frac{d\log L(\Theta)}{d\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} x_i + (\gamma-1)\sum_{i=1}^{n} \frac{e^{-\beta x_i} x_i}{1-e^{-\beta x_i}} - 2\sum_{i=1}^{n} \frac{\gamma e^{-\beta x_i} x_i \left(1-e^{-\beta x_i}\right)^{\gamma-1}}{1-(1-e^{-\beta x_i})^{\gamma}} = 0 \tag{43}$$

By setting the nonlinear system of equations to zero, that is, $\frac{d\log L(\Theta)}{d\alpha} = \frac{d\log L(\Theta)}{d\lambda} = \frac{d\log L(\Theta)}{d\gamma} = \frac{d\log L(\Theta)}{d\beta} = 0$ and solving them simultaneously gives the MLE$\Theta = (\alpha, \lambda, \gamma, \beta)^T$. Moreover, numerical computations are needed to find the solution of this nonlinear system of equations, $\frac{d\log L(\Theta)}{d\alpha} = \frac{d\log L(\Theta)}{d\lambda} = \frac{d\log L(\Theta)}{d\gamma} = \frac{d\log L(\Theta)}{d\beta} = 0$ and it is usually more convenient to obtain by using nonlinear optimization methods such as the Newton-Rapson algorithm.

Asymptotic Confidence Intervals

In the asymptotic scenario, the distribution of the maximum likelihood estimators (MLE) $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\beta}$ tends to be approximately multivariate normal. Therefore, as the sample size, n, approaches infinity (as $n \to \infty$), the asymptotic distribution of the MLE $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\beta}$ is described by

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$$\begin{pmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\gamma}\\ \hat{\beta}\end{pmatrix} \sim N\left[\begin{pmatrix}\hat{\alpha}\\ \hat{\lambda}\\ \hat{\gamma}\\ \hat{\beta}\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} & \hat{V}_{14}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} & \hat{V}_{24}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} & \hat{V}_{34}\\ \hat{V}_{41} & \hat{V}_{42} & \hat{V}_{43} & 44\end{pmatrix}\right]$$

where

$$V_{ij}^{-1} = -E\begin{bmatrix}\frac{\partial^2 L(\Theta)}{\partial\alpha^2} & \frac{\partial^2 L(\Theta)}{\partial\alpha\,\partial\lambda} & \frac{\partial^2 L(\Theta)}{\partial\alpha\partial\gamma} & \frac{\partial^2 L(\Theta)}{\partial\alpha\partial\beta}\\ \frac{\partial^2 L(\Theta)}{\partial\lambda\partial\alpha} & \frac{\partial^2 L(\Theta)}{\partial\lambda^2} & \frac{\partial^2 L(\Theta)}{\partial\lambda\partial\gamma} & \frac{\partial^2 L(\Theta)}{\partial\lambda\partial\beta}\\ \frac{\partial^2 L(\Theta)}{\partial\gamma\partial\alpha} & \frac{\partial^2 L(\Theta)}{\partial\gamma\partial\lambda} & \frac{\partial^2 L(\Theta)}{\partial\gamma^2} & \frac{\partial^2 L(\Theta)}{\partial\gamma\partial\beta}\\ \frac{\partial^2 L(\Theta)}{\partial\beta\partial\alpha} & \frac{\partial^2 L(\Theta)}{\partial\beta\partial\lambda} & \frac{\partial^2 L(\Theta)}{\partial\beta\partial\gamma} & \frac{\partial^2 L(\Theta)}{\partial\beta^2}\end{bmatrix};\ i,j = 1,2,3,4 \qquad (44)$$

Solving the inverse matrix of the observed information matrix in equation (44) gives the asymptotic variance and co-variance of the maximum likelihood estimators $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$, and $\hat{\beta}$. The approximate $(1-\alpha)100\%$ asymptotic confidence intervals (C. I) for $\hat{\alpha}, \hat{\lambda}, \hat{\gamma}$, and $\hat{\beta}$ are given by

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}\ ; \quad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}\ ; \quad \hat{\gamma} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}}\ ; \quad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{44}}$$

where $Z_{\frac{\alpha}{2}}$ is the upper αth percentile of the standard normal distribution.

V. Simulation Study

In this section the simulation study is conducted using the quantile function of the OEHL-EE distribution in equation 45 with the help of R-statistics package. Table 1 – 4 presents results for the evaluation of the effect of each parameter of the OEHL-EE model on the median, mean, variance, coefficient of variation (CV), skewness and kurtosis of the model. In this simulation study, sample size of 1000 was generated and it was done using some arbitrary values of each parameters of the OEHL-EE model. Figures 5,6,7 and 8 respectively was also used to illustrate the median, mean, variance, coefficient of variation (CV), skewness and kurtosis of the model

$$x_U = -\frac{1}{\beta}\ln\left\{1 - \left(\frac{-\ln\left(1-u^{\frac{1}{\alpha}}\right) + \ln\left(1+u^{\frac{1}{\alpha}}\right)}{\lambda - \ln\left(1-u^{\frac{1}{\alpha}}\right) + \ln\left(1+u^{\frac{1}{\alpha}}\right)}\right)^{\frac{1}{\gamma}}\right\} \qquad (45)$$

where, u has uniform distribution over the interval (0, 1); i.e., $u \sim U(0,1)$.

Table 1: Median, Mean, Variance, CV, Skewness and Kurtosis for some selected values of shape parameter α, of the OEHL-EE model while the parameters $\lambda = 10,\ \gamma = 2$ and $\beta = 0.02$ are fixed.

α	Min.	Median	Max.	Mean	Variance	STD. DEV.	CV	Coeff. Skewness	Coeff. Kurtosis
0.5	0.0099	12.6733	48.868	13.9003	92.1828	9.6012	0.6907	0.3011	0.0968
1	0.4738	19.0864	51.5417	19.4833	85.8451	9.2653	0.4756	0.3021	0.1163
1.5	1.7356	22.443	53.0344	22.6953	77.0676	8.7788	0.3868	0.3047	0.1183
2	3.3504	24.6482	54.0642	24.879	70.2491	8.3815	0.3369	0.3054	0.2021

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2.5	4.9982	26.2677	54.8471	26.5055	65.0971	8.0683	0.3044	0.3082	0.2633
2.6	5.3180	26.5447	54.9834	26.7851	64.2207	8.0138	0.2992	0.3100	0.2734
2.7	5.6332	26.8092	55.1141	27.0522	63.3875	7.9616	0.2943	0.3120	0.2829
3	6.5477	27.5372	55.4770	27.7888	61.1172	7.8177	0.2813	0.3189	0.3081
3.5	7.9594	28.5758	56.0028	28.8416	57.9567	7.6129	0.2640	0.3321	0.3421
3.7	8.4844	28.9428	56.1909	29.2141	56.8668	7.5410	0.2581	0.3376	0.3535
3.8	8.7386	29.1176	56.2809	29.3915	56.3533	7.5069	0.2554	0.3403	0.3589
4	9.2311	29.4514	56.4534	29.7304	55.3830	7.4420	0.2503	0.3457	0.3689
4.5	10.3739	30.2063	56.8473	30.4969	53.2415	7.2967	0.2393	0.3588	0.3908
5	11.4032	30.8683	57.1967	31.1691	51.4269	7.1713	0.2301	0.3710	0.4093
6	13.1787	31.9861	57.7953	32.3034	48.5043	6.9645	0.2156	0.3927	0.4393
7	14.6576	32.9052	58.2953	33.2354	46.2363	6.7997	0.2046	0.4111	0.4631
10	17.9434	34.9488	59.4320	35.3035	41.6233	6.4516	0.1827	0.4519	0.5144
12	19.5349	35.9524	60.0024	36.3170	39.5641	6.2900	0.1732	0.4713	0.5391
14	20.8318	36.7810	60.4793	37.1527	37.9592	6.1611	0.1658	0.4868	0.5594
16	21.9197	37.4848	60.8884	37.8618	36.6601	6.0548	0.1599	0.4996	0.5765
18	22.8528	38.0953	61.2464	38.4764	35.5784	5.9648	0.1550	0.5103	0.5913
20	23.6672	38.6335	61.5643	39.0178	34.6581	5.8871	0.1509	0.5195	0.6042

Source: Simulated data of OEHL-EE

The graphical representations in Figure 5 complement the numerical results in Table 1.

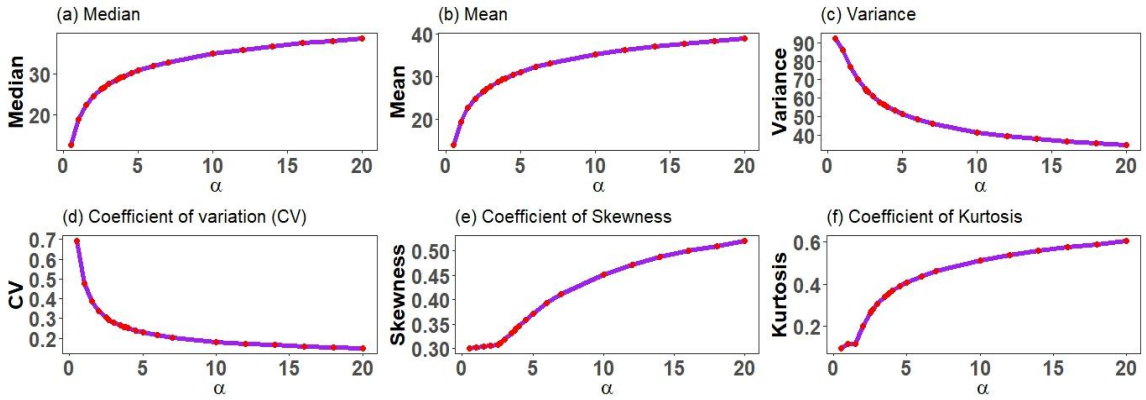


Figure 5: Plots of the median, mean, variance, CV, skewness, and kurtosis of the OEHL-EE distribution against $\boldsymbol{\alpha}$ when the parameters $\boldsymbol{\lambda = 10,\ \gamma = 2}$ and $\boldsymbol{\beta = 0.02}$ are fixed

Table 2: Median, Mean, Variance, CV, Skewness and Kurtosis for some selected values of shape parameter λ, of the OEHL-EE model while the parameters $\alpha = 12,\ \gamma = 9$ and $\beta = 0.03$ fixed

λ	Min.	Median	Max.	Mean	Variance	STD. DEV.	CV	Coef. Skewness	Coef. Kurtosis
0.5	115.9401	141.2337	174.9189	141.6542	98.8018	9.9399	0.0702	0.1875	-0.0266
1	97.1414	120.3335	152.6619	120.8024	87.3073	9.3438	0.0773	0.2324	0.0024
1.5	87.2516	108.8248	139.9629	109.3223	78.3788	8.8532	0.0810	0.267	0.0309
2	80.8016	101.0787	131.1608	101.5928	71.2277	8.4396	0.0831	0.2945	0.0575
2.5	76.1367	95.3464	124.483	95.8695	65.3628	8.0847	0.0843	0.3168	0.0816
3	72.5462	90.8575	119.1409	91.3845	60.4610	7.7757	0.0851	0.3352	0.1035
3.5	69.6642	87.2061	114.7145	87.7337	56.2999	7.5033	0.0855	0.3506	0.1231
4	67.2799	84.1534	110.9535	84.6793	52.7214	7.2609	0.0857	0.3636	0.1406
4.5	65.2614	81.5474	107.6971	82.0702	49.6097	7.0434	0.0858	0.3748	0.1564
5	63.5216	79.2860	104.8355	79.8047	46.8781	6.8468	**0.0858**	0.3843	0.1705

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7	58.3641	72.5087	96.0507	73.0057	38.6228	6.2147	**0.0851**	0.4118	0.2144
18	46.7126	56.9550	74.7117	57.3470	20.8464	4.5658	**0.0796**	0.4541	0.2982
20	45.6313	55.5094	72.6573	55.8879	19.3925	4.4037	0.0788	0.4554	0.3025
24	43.8476	53.1292	69.2576	53.4847	17.1052	4.1358	0.0773	0.4564	0.3074
26	43.0978	52.1308	67.8263	52.4764	16.1878	4.0234	0.0767	**0.4564**	0.3086
30	41.8051	50.4133	65.359	50.7414	14.6708	3.8303	0.0755	**0.4555**	**0.3094**
33	40.9766	49.3155	63.7795	49.6322	13.7432	3.7072	0.0747	**0.4545**	**0.3091**
35	40.4776	48.6554	62.8293	48.9652	13.2016	3.6334	0.0742	0.4537	**0.3086**
37	40.0148	48.0441	61.9491	48.3475	12.7110	3.5653	0.0737	0.4528	0.3078
39	39.5837	47.4755	61.1305	47.7730	12.2644	3.5021	0.0733	0.4519	0.307
40	39.3789	47.2057	60.742	47.5003	12.0556	3.4721	0.0731	0.4514	0.3065

Source: Simulated data of OEHL-EE

The graphical representations in Figure 2 complement the numerical results in Table 2.

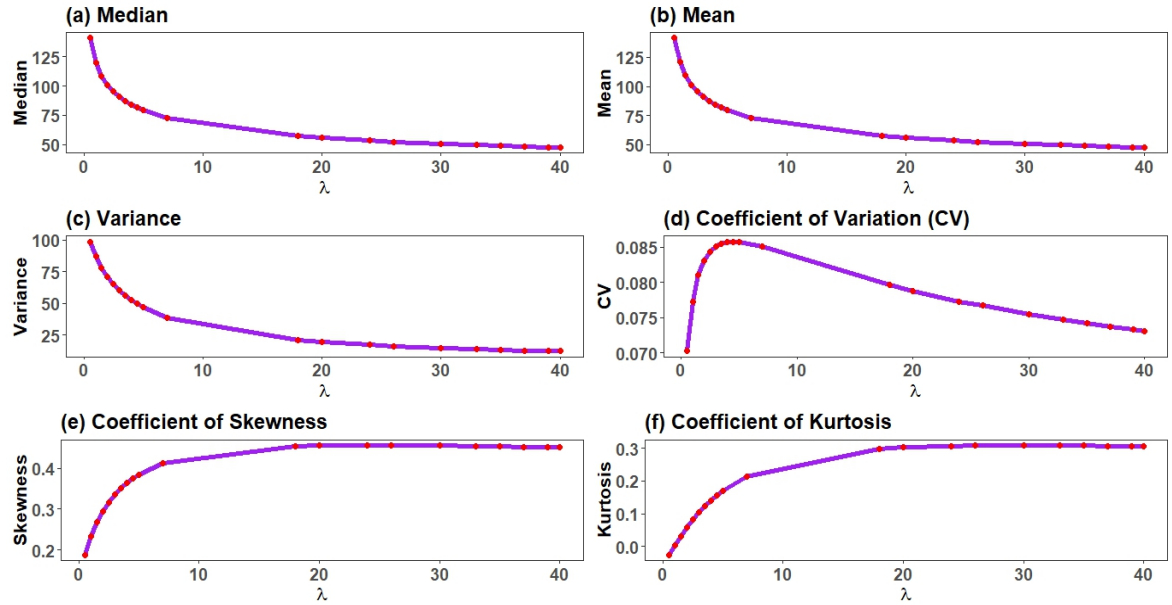


Figure 6: Plots of the median, mean, variance, CV, skewness, and kurtosis of the OEHL-EE distribution against $\boldsymbol{\lambda}$ when $\boldsymbol{\alpha = 12,\ \gamma = 9}$ and $\boldsymbol{\beta = 0.03}$ fixed

Table 3: Median, Mean, Variance, CV, Skewness and Kurtosis for some selected values of shape parameter γ, of the OEHL-EE model while the parameters $\alpha = 7,\ \lambda = 15$ and $\beta = 0.04$ are fixed

γ	Min.	Median	Max.	Mean	Variance	STD. DEV.	CV	Coef. Skewness	Coef. Kurtosis
0.5	0.0810	0.7313	4.0043	0.8652	0.3112	0.5579	0.6448	1.5682	3.5177
1	1.4635	4.6520	12.1415	4.8831	2.8555	1.6898	0.3461	0.7884	0.8459
1.5	4.0004	9.1548	18.8203	9.3976	5.7011	2.3877	0.2541	0.5687	0.4246
2	6.8102	13.2783	24.2061	13.5181	7.9492	2.8194	0.2086	0.4673	0.2803
2.5	9.5547	16.9320	28.6758	17.1665	9.6568	3.1075	0.1810	0.4094	0.2120
3	12.1345	20.1714	32.4842	20.4008	10.9709	3.3122	0.1624	0.3720	0.1734
3.5	14.5308	23.0658	35.7973	23.2907	12.0046	3.4648	0.1488	0.3460	0.1490
3.8	15.8833	24.6622	37.5957	24.8847	12.5236	3.5389	0.1422	0.3338	0.1383
4	16.7513	25.6749	38.7273	25.8960	12.8354	3.5827	0.1383	0.3268	0.1324
5	20.7293	30.2188	43.7303	30.4339	14.0834	3.7528	0.1233	0.3005	0.1113
7	27.2602	37.4281	51.4783	37.6354	15.6385	3.9545	0.1051	0.2711	0.0903

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10	34.7300	45.4291	59.8934	45.6299	16.9007	4.1110	0.0901	0.2496	0.0765
14	42.1598	53.2237	67.9687	53.4198	17.7902	4.2178	0.0790	0.2356	0.0683
18	47.8910	59.1613	74.0639	59.3547	18.3016	4.2780	0.0721	0.2278	0.0640
22	52.5545	63.9575	78.961	64.1491	18.6336	4.3167	0.0673	0.2230	0.0614
26	56.4852	67.9808	83.0544	68.1712	18.8664	4.3436	0.0637	0.2196	0.0596
30	59.8821	71.4459	86.5711	71.6353	19.0388	4.3633	0.0609	0.2171	0.0584
33	62.1579	73.7621	88.9178	73.9510	19.1412	4.3751	0.0592	0.2157	0.0576
37	64.9024	76.5504	91.7391	76.7386	19.2525	4.3878	0.0572	0.2141	0.0568
40	66.7797	78.4548	93.6641	78.6427	19.3217	4.3956	0.0559	0.2132	0.0564
48	71.1900	82.9211	98.1725	83.1083	19.4645	4.4119	0.0531	0.2112	0.0554

Source: Simulated data of OEHL-EE

The graphical representations in Figure 7 complement these numerical results in Table 3.

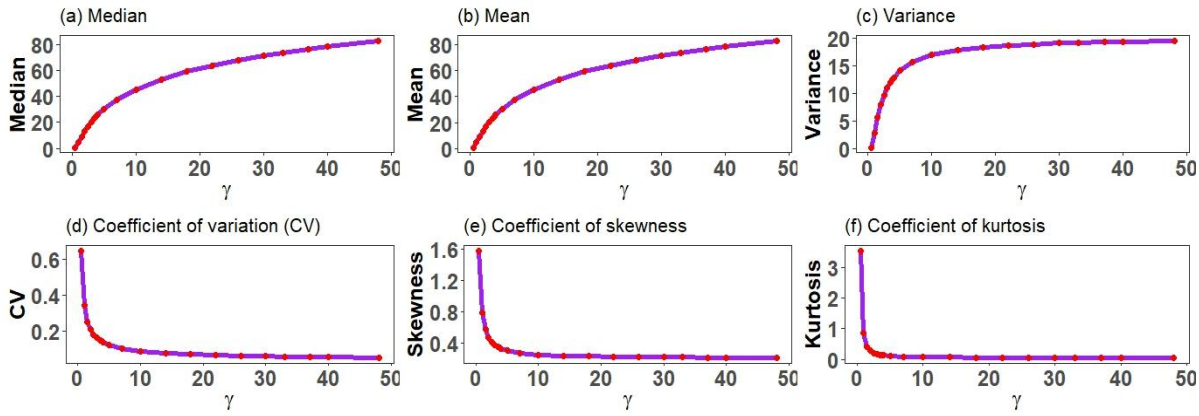


Figure 7: Plots of the median, mean, variance, CV, skewness, and kurtosis of the OEHL-EE distribution against $\boldsymbol{\gamma}$ when $\boldsymbol{\alpha = 7,\ \lambda = 15}$ and $\boldsymbol{\beta = 0.04}$ are fixed

Table 4: Median, Mean, Variance, CV, Skewness and Kurtosis for some selected values of shape parameter β, of the OEHL-EE model while the parameters $\alpha = 8,\ \lambda = 20$ and $\gamma = 16$ are fixed

$\boldsymbol{\beta}$	Min.	Median	Max.	Mean	Variance	STD. DEV.	CV	Coef. Skewness	Coef. Kurtosis
0.01	158.6972	214.6617	269.8424	215.0189	230.1314	15.1701	0.0705524	0.1831	0.0299
0.02	79.3486	107.3309	134.9212	107.5094	57.5328	7.5850	0.0705520	0.1831	0.0299
0.03	52.8991	71.5539	89.9475	71.6730	25.5702	5.0567	0.0705524	0.1831	0.0299
0.04	39.6743	53.6654	67.4606	53.7547	14.3832	3.7925	0.0705520	0.1831	0.0299
0.05	31.7394	42.9323	53.9685	43.0038	9.2053	3.034	0.0705519	0.1831	0.0299
0.06	26.4495	35.777	44.9737	35.8365	6.3925	2.5283	0.0705510	0.1831	0.0299
0.07	22.671	30.666	38.5489	30.7170	4.6966	2.1672	0.0705538	0.1831	0.0299
0.08	19.8372	26.8327	33.7303	26.8774	3.5958	1.8963	0.0705537	0.1831	0.0299
0.09	17.633	23.8513	29.9825	23.8910	2.8411	1.6856	0.0705538	0.1831	0.0299
0.1	15.8697	21.4662	26.9842	21.5019	2.3013	1.5170	0.0705519	0.1831	0.0299
0.2	7.9349	10.7331	13.4921	10.7509	0.5753	0.7585	0.0705522	0.1831	0.0299
0.3	5.2899	7.1554	8.9947	7.1673	0.2557	0.5057	0.0705566	0.1831	0.0299
0.4	3.9674	5.3665	6.7461	5.3755	0.1438	0.3793	0.0705609	0.1831	0.0299
0.5	3.1739	4.2932	5.3968	4.3004	0.0921	0.3034	0.0705516	0.1831	0.0299
0.6	2.645	3.5777	4.4974	3.5836	0.0639	0.2528	0.0705436	0.1831	0.0299
0.7	2.2671	3.0666	3.8549	3.0717	0.047	0.2167	0.0705473	0.1831	0.0299
0.8	1.9837	2.6833	3.373	2.6877	0.036	0.1896	0.0705436	0.1831	0.0299
0.9	1.7633	2.3851	2.9982	2.3891	0.0284	0.1686	0.0705705	0.1831	0.0299

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1	1.587	2.1466	2.6984	2.1502	0.023	0.1517	0.0705516	0.1831	0.0299
2	0.7935	1.0733	1.3492	1.0751	0.0058	0.0759	0.0705981	0.1831	0.0299

Source: Simulated data of OEHL-EE

The graphical representations in Figure 8 complement the numerical results in Table 4.

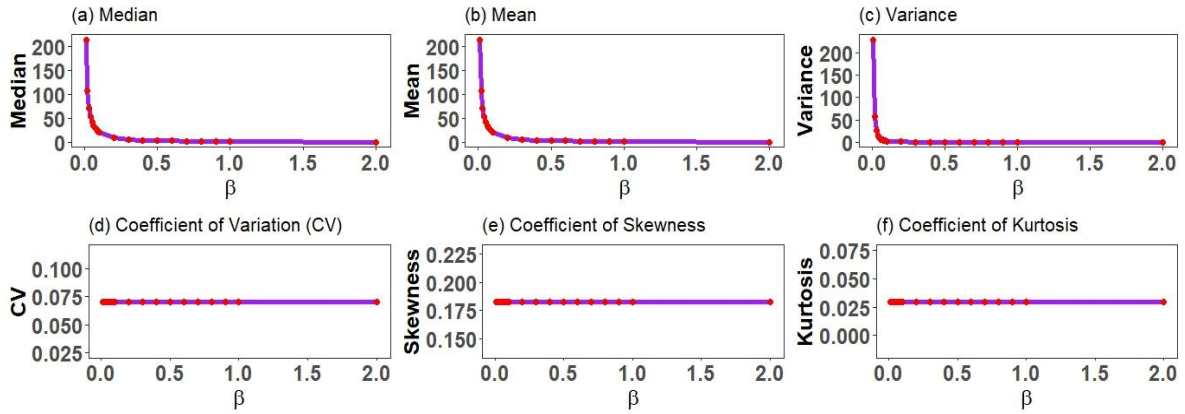


Figure 8: Plots of the median, mean, variance, CV, skewness, and kurtosis of the OEHL-EE distribution against $\boldsymbol{\beta}$ when $\boldsymbol{\alpha = 8,\ \lambda = 20}$ and $\boldsymbol{\gamma = 16}$ are fixed

Table 1, 2,3 and 4, it is observed that both the **mean** and **median** increase gradually as the α increases. This monotonic increase shows that the distribution is more right skewed at larger values of α. This implies that there is higher mean and median heavier tails as x increases. The **variance** and **standard deviation** show a decreasing trend with increasing α values. This inverse relationship shows that as α increases, the variation of the distribution becomes narrower. This shows that there is more concentration of probability mass around the mean. The **coefficient of skewness** shows that the OEHL-EE distribution shows a **positively skewed** behaviour for all selected values of α. This result indicates that the distribution keeps a longer right tail for all the selected values of α and it implies that there is an increase in asymmetry as α increases. **The finding shows that increasing α value decreases variability in the distribution and it makes the mean and median, skewness, and kurtosis values to increase.** These behaviours show the **flexibility of the OEHL-EE distribution** in modeling data with different degrees of asymmetry and tail thickness. The flexibility of OEHL-EE distribution makes it suitable for modeling a wide range of real-life data from various areas.

Table 5: The ML estimates, biases, variances and MSEs of the OEHL-EE distribution for some selected values of $= 2.5,\ \lambda = 1.2,\ \gamma = 1.8$, and $\beta = 0.5$

Sample size (n)	Parameter	ML Estimate	$\|Bias\|$	Variance	MSE
20	$\hat{\alpha}$	0.3988	2.1012	0.0427	4.4577
	$\hat{\lambda}$	0.1223	1.0777	0.1070	1.2684
	$\hat{\gamma}$	0.9137	0.8863	4.2320	5.0175
	$\hat{\beta}$	1.7537	1.2537	0.8533	2.4251
50	$\hat{\alpha}$	1.7428	0.7572	2.3368	2.9102
	$\hat{\lambda}$	0.1202	1.0798	0.1009	1.2669
	$\hat{\gamma}$	0.9222	0.8778	4.2300	5.0005
	$\hat{\beta}$	0.8953	0.3953	0.1448	0.3011
80	$\hat{\alpha}$	1.7451	0.7549	2.2472	2.8171
	$\hat{\lambda}$	0.7644	0.4356	1.1096	1.2993
	$\hat{\gamma}$	0.9407	0.8593	2.8744	3.6128
	$\hat{\beta}$	0.3210	0.1790	0.0173	0.0493

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Sample size (n)	Parameter	ML Estimate	$\|Bias\|$	Variance	MSE
120	$\hat{\alpha}$	2.0425	0.4575	1.4021	1.6114
	$\hat{\lambda}$	0.2162	0.9838	0.2674	1.2353
	$\hat{\gamma}$	1.3301	0.4699	2.3120	2.5328
	$\hat{\beta}$	0.5241	0.0241	0.0162	0.0168
200	$\hat{\alpha}$	2.3392	0.1608	0.3326	0.3585
	$\hat{\lambda}$	1.7921	0.5921	0.4001	0.7507
	$\hat{\gamma}$	1.6005	0.1995	1.2211	1.2609
	$\hat{\beta}$	0.3254	0.1746	0.1716	0.2021
300	$\hat{\alpha}$	3.0903	0.5903	1.1011	1.4496
	$\hat{\lambda}$	1.6274	0.4274	1.1083	1.2910
	$\hat{\gamma}$	1.7612	0.0388	0.1418	0.1433
	$\hat{\beta}$	0.6659	0.1659	0.0212	0.0487
500	$\hat{\alpha}$	3.2338	0.7338	1.7181	2.2566
	$\hat{\lambda}$	1.0071	0.1929	1.3878	1.4250
	$\hat{\gamma}$	2.0172	0.2172	1.2435	1.2907
	$\hat{\beta}$	0.5677	0.0677	0.0058	0.0104

Source: Simulated data of OEHL-EE

Table 6: The ML estimates, biases, variances and MSEs of the OEHL-EE distribution for the some selected values of $= 7,\ \lambda = 4,\ \gamma = 5$, and $\beta = 0.9$

Sample size (n)	Parameter	ML Estimate	$\|Bias\|$	Variance	MSE
20	$\hat{\alpha}$	3.3220	3.6780	6.9207	20.4484
	$\hat{\lambda}$	1.4993	2.5007	5.2866	11.5401
	$\hat{\gamma}$	3.6097	1.3903	3.7374	5.6703
	$\hat{\beta}$	1.8253	0.9253	3.6288	4.4850
50	$\hat{\alpha}$	3.3700	3.6300	5.6595	18.8364
	$\hat{\lambda}$	2.1562	1.8438	3.0401	6.4397
	$\hat{\gamma}$	4.5308	0.4692	1.3621	1.5822
	$\hat{\beta}$	1.1280	0.2280	0.4837	0.5357
80	$\hat{\alpha}$	3.6188	3.3812	3.9382	15.3707
	$\hat{\lambda}$	2.4173	1.5827	2.0175	4.5224
	$\hat{\gamma}$	4.8365	0.1635	0.8189	0.8456
	$\hat{\beta}$	0.4803	0.4197	0.2604	0.4365
120	$\hat{\alpha}$	4.1762	2.8238	3.0366	11.0104
	$\hat{\lambda}$	3.0277	0.9723	1.9348	2.8802
	$\hat{\gamma}$	5.5038	0.5038	1.1928	1.4466
	$\hat{\beta}$	1.0063	0.1063	0.2164	0.2277
200	$\hat{\alpha}$	4.2456	2.7544	3.2941	10.8808
	$\hat{\lambda}$	4.6849	0.6849	1.4850	1.9541
	$\hat{\gamma}$	5.5347	0.5347	1.5726	1.8585
	$\hat{\beta}$	0.3851	0.5149	0.1391	0.4042
300	$\hat{\alpha}$	4.8012	2.1988	2.0642	6.8989
	$\hat{\lambda}$	5.3930	1.3930	2.6608	4.6012
	$\hat{\gamma}$	6.3770	1.3770	2.1653	4.0614
	$\hat{\beta}$	1.0024	0.1024	0.0833	0.0938
500	$\hat{\alpha}$	6.4979	0.5021	1.5627	1.8148

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	$\hat{\lambda}$	5.5627	1.5627	2.9940	5.4360
	$\hat{\gamma}$	8.0663	3.0663	3.0118	12.4140
	$\hat{\beta}$	0.9188	0.0188	0.1099	0.1103

Source: Simulated data of OEHL-EE

Discussion of the Simulation Results

Table 5 and 6 show the results from simulation study. The Monte Carlo simulation technique is used to generate samples of 20, 50, 80, 120, 200, 300, and 500 for different combinations of parameters $(\alpha, \lambda, \gamma, \beta)$ of OEHL-EE distribution. The simulation is repeated 1000 times for each sample size. The purpose of this simulation study is to examine the performance of the ML estimator in the estimation of parameter of the proposed OEHL-EE distribution and evaluation of the estimator is based on its mean, the absolute bias and the mean squared error (MSE) of the estimates. The estimates were obtained using optim function in R version 4.5.1 (R Core Team, 2025). As it can be seen in Table 5 and 6, the results show that as the sample size increases, the ML estimates tend to converge towards their true parameter values and this indicates the consistency of the estimator.

The findings from this simulation study in Table 5 and 6 shows that both the bias and MSE decrease as the sample size increases and this implies that the ML estimator is asymptotically unbiased and efficient. The results provide strong empirical evidence that the ML estimator is suitable for estimating the parameters of the OEHL-EE distribution especially when adequate sample sizes are available.

VI. Applications

Application to a Real Life Dataset

The potentiality of OEHL-EE distribution is illustrated in this section through the application of two real-life data sets. The OEHL-EE distribution is compared with other competing distributions; Weibull (W) (Weibull, 1951), Gamma (G) (Stacy,1962; Johnson, 1996), Kumaraswamy-Weibull (Kum-W) (Cordeiro et al., 2010), Odd Exponentiated Half Logistic-Weibull (OEHL-W) (Afify et al., 2017), Odd Exponentiated Half Logistic-Gamma (OEHL-Ga) (Afify et al., 2017), Odd Exponentiated Half Logistic-Normal (OEHL-N) (Afify et al., 2017), Exponentiated Exponential (EE) (Gupta and Kundu, 2001), Transmuted Exponential-Weibull (TE-W) (Mohammed and Ugwuowo, 2020), New Exponential-Exponential (NE-E) (Ogunwale et al., 2022), Inverse Power Chi-Square (IPCS) (Ishaq et al., 2023), and Half Logistic Exponentiated Inverted Weibull (HLEIW) (Oseghale, et al., 2025).

The maximum likelihood estimates of the parameters of the distributions are obtained (see Section 3.28 for their pdfs).

Tables 8 and 11 gives the MLEs of the unknown parameter(s) and Tables 9 and 12 gives the goodness of fit measures log-likelihood (*LL*), Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A*), Cramér-von Mises (W*) and Kolmogorov-Smirnov (K-S) for the fitted lifetime distribution.

First Real-life Dataset I

The objective in this section is to use a real-life dataset to demonstrate that the proposed OEHL-EE model offers a superior fit to real-life data when compared to other competing models. The data set represents the monthly inflation rate in Nigeria from January 2003 to June 2023. This dataset has been used by Ishaq et al. (2023). The data are as follows: 10.6, 7.3, 5.9, 8.3, 8.7, 14, 12.9, 12.4, 18.4, 23.6, 21.3, 23.8, 22.4, 24.8, 22.5, 17.5, 19.8, 14.1, 10.7, 13, 9.1, 10.7, 10, 10, 9.8, 10.9, 16.3, 17.9, 16.8, 18.6, 26.1, 28.2, 24.3, 18.6, 15.1, 11.6, 10.7, 10.8, 12, 12.6, 10.5, 8.5, 3, 3.7, 6.3, 6.1, 7.8, 8.5, 8, 7.1, 5.2, 4.2, 4.6, 6.4, 4.8, 4.2, 4.1, 4.6, 5.2, 6.6, 8.6, 8, 7.8, 8.2, 9.7, 12, 14, 12.4, 13, 14.7, 14.8, 15.1, 14, 14.6, 14.4, 13.3, 13.2, 11.2, 11.1, 11, 10.4, 11.6, 12.4, 13.9, 14.4, 15.6, 14.8, 15, 12.9, 14.1, 13, 13.7, 13.6, 13.4, 12.8, 11.8, 12.1, 11.1, 12.8, 11.3, 12.4, 10.2, 9.4, 9.3, 10.3, 10.5, 10.5, 10.3, 12.6, 11.9, 12.1, 12.9, 12.7, 12.9, 12.8, 11.7, 11.3, 11.7, 12.3, 12, 9, 9.5, 8.6, 9.1, 9, 8.4, 8.7, 8.2, 8, 7.8, 7.9, 8, 8, 7.7, 7.8, 7.9, 8, 8.2, 8.3, 8.5, 8.3, 8.1, 7.9, 8, 8.2, 8.4, 8.5, 8.7, 9, 9.2, 9.2, 9.3, 9.4, 9.3, 9.37, 9.55, 9.62, 11.38, 12.77, 13.72, 15.58, 16.48, 17.13, 17.61, 17.85, 18.33, 18.48, 18.55, 18.72, 17.78, 17.26, 17.24, 16.25, 16.1, 16.05, 16.01, 15.98, 15.91, 15.9, 15.37, 15.13, 14.33, 13.34, 12.48, 11.61, 11.23, 11.14, 11.23, 11.28, 11.26, 11.28, 11.44, 11.37, 11.31, 11.25, 11.37, 11.4, 11.22, 11.08, 11.02, 11.24, 11.61,11.85, 11.98, 12.13,

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12.2, 12.26, 12.34, 12.4, 12.56, 12.82, 13.22, 13.71, 14.23, 14.89, 15.75, 16.47, 17.33, 18.17, 18.12, 17.93, 17.75, 17.38, 17.01, 16.63, 15.99, 15.4, 15.63, 15.6, 15.7, 15.92, 16.82, 17.71, 18.6, 19.64, 20.52, 20.77, 21.09, 21.47, 21.34, 21.82, 21.91, 22.04, 22.22, 22.41, 22.79.

Second Real-life Dataset II

The real life-time dataset to use in this study are secondary data. The data set represents 76 observations of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. These are complete data with the exact times of failure. The data had been used by Owoloko et al. (2015), Jayakumar and Babu (2017), Mohammed and Ugwuowo (2020). This data set is used to compare the OEHL-EE model with other aforementioned models.

The data are: 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 7: Descriptive statistics for dataset I and II

	Min.	Q1	Median	Mean	Q3	Max.	Variance	Skewness	Kurtosis
Dataset I	3.0000	9.3770	12.1650	12.8428	15.7380	28.2000	21.1564	0.5839	0.1999
Dataset II	0.0252	0.9048	1.7362	1.9592	2.2959	9.0960	2.4774	1.9406	4.9474

Table 7 shows that the two datasets are positively skewed and the coefficient of skewness for dataset I is 0.5839 and dataset II is 1.9406. It also shows that the mean values for dataset I and dataset II are 12.8428 and 1.9592, respectively and they are greater than their median values which are 12.1650 for dataset I and 1.7362 for dataset II. The results from Table 7 show clearly that the datasets are asymmetric and positively skewed. The data are analysed using R software version 4.5.1; and Table 8 and 9 present the computed maximum likelihood estimates and goodness-of-fit measures obtained through this analysis.

Table 8: Estimated Parameters and Goodness-of-Fit for Dataset I (the monthly inflation rate in Nigeria from January 2003 to June 2023)

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
OEHL-EE	$\hat{\alpha} = 3.423$ $\hat{\lambda} = 8.3492$ $\hat{\gamma} = 1.3178$ $\hat{\beta} = 0.0302$	**716.0317**	**1440.063**	**1432.097**	**1445.709**	**1438.886**	**0.0888**	**0.5959**	**0.0523**
OEHL-W	$\hat{\theta} = 0.8050$ $\hat{\lambda} = 50.000$ $\hat{a} = 3.0292$ $\hat{b} = 50.000$	742.2923	1492.585	1484.618	1498.230	1491.408	0.7721	4.033	0.1255
OEHL-Ga	$\hat{\theta} = 3.2969$ $\hat{\lambda} = 3.7694$ $\hat{a} = 1.1965$ $\hat{b} = 20.000$	716.6503	1441.301	1433.334	1446.946	1440.123	0.1234	0.7838	0.0618
OEHL-N	$\hat{\alpha} = 8.1429$ $\hat{\lambda} = 8.7072$ $\hat{\mu} = 25.2573$ $\hat{\sigma} = 20.000$	723.1112	1454.222	1446.256	1459.868	1453.045	0.2932	1.8701	0.0841

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EE	$\hat{\gamma} = 12.2120$ $\hat{\beta} = 0.248$	721.1969	1446.394	14442.41	1449.217	1445.805	0.1423	1.1105	0.0666
Kum-W	$\hat{a} = 1.5738$ $\hat{b} = 1.0237$ $\hat{c} = 2.0765$ $\hat{\lambda} = 0.0822$	721.3913	1450.783	1442.816	1456.428	1449.605	0.3247	2.2083	0.1032
TE-W	$\hat{\lambda} = 6.1216$ $\hat{k} = 3.2055$ $\hat{\gamma} = 28.6038$ $\hat{\theta} = 0.6144$	718.8436	1445.687	1437.720	1451.333	1444.51	0.1964	1.2059	0.0734
HLEIW	$\hat{\alpha} = 20.000$ $\hat{\theta} = 0.7741$ $\hat{\rho} = 20.000$	721.4042	1448.808	1442.833	1453.043	1447.926	0.3370	2.1554	0.0960
W	$\hat{\beta} = 2.9794$ $\hat{\eta} = 14.3861$	721.4028	1446.806	1442.822	1449.629	1446.217	0.2843	1.7221	0.0831
G	$\hat{\alpha} = 7.5832$ $\hat{\beta} = 0.5905$	716.6801	1437.360	1433.377	1440.183	1436.772	0.0789	0.5884	0.05354
IPCS	$\hat{\alpha} = 10$ $\hat{\beta} = 100$	170241.7	340487.4	340483.4	340490.2	340486.8	50.616	469.91	0.6631

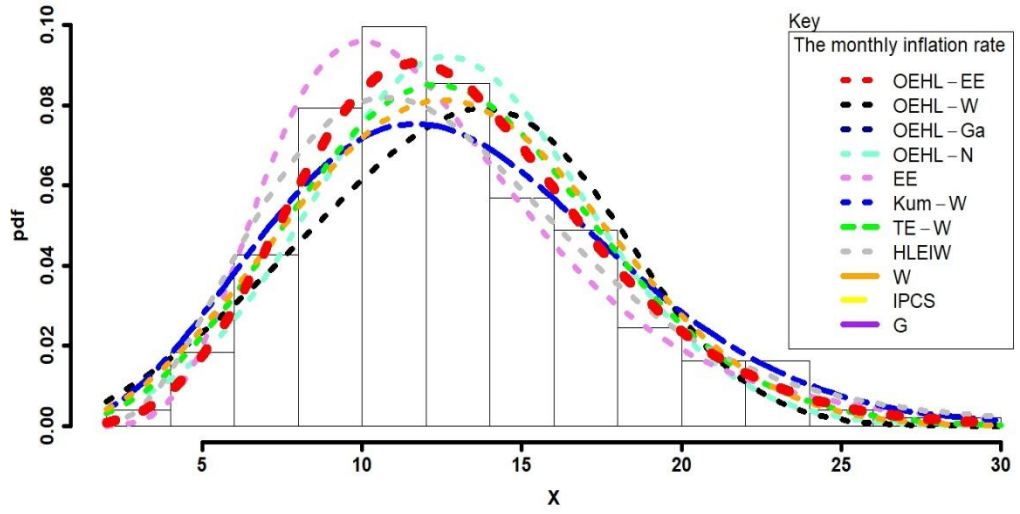


Figure 9: Plot of pdf for the estimates of the fitted models to dataset I (the monthly inflation rate)

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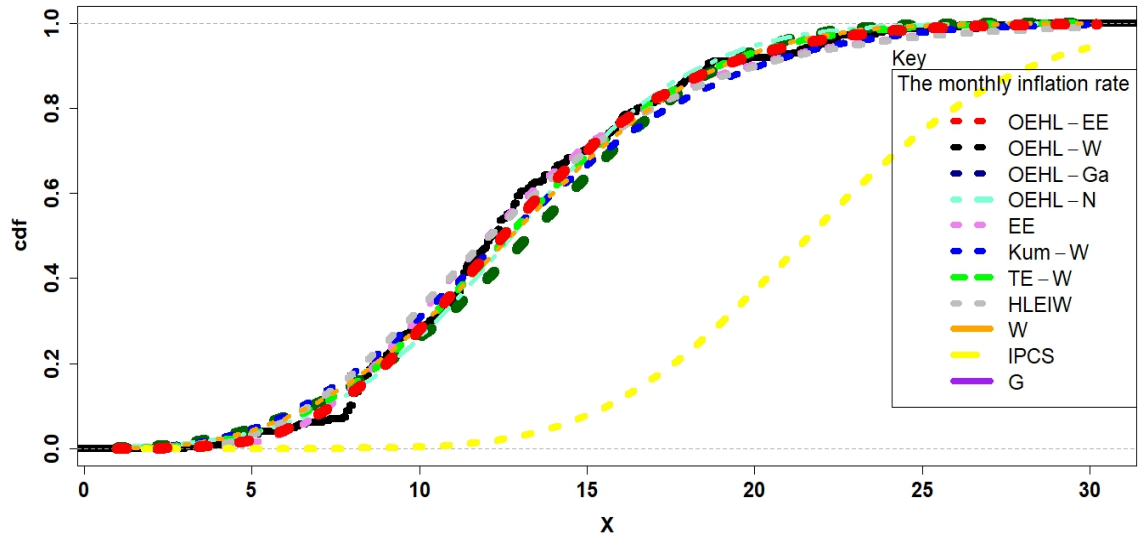


Figure 10: Plot of cdf of the fitted models and empirical cdf to dataset I (the monthly inflation rate)

Table 9: Estimated Parameters and Goodness-of-Fit for Dataset II (life of fatigue fracture of Kevlar 373/epoxy)

Model	ML Estimates	-LL	AIC	CAIC	HQIC	BIC	W^*	A^*	D^*
OEHL-EE	$\hat{\alpha} = 3.5173$ $\hat{\lambda} = 3.7055$ $\hat{\gamma} = 0.3781$ $\hat{\beta} = 0.0523$	**121.9462**	**251.8923**	**244.0050**	**255.6182**	**249.7553**	**0.1053**	**0.6292**	**0.1005**
OEHL-W	$\hat{\theta} = 1.0426$ $\hat{\lambda} = 50.000$ $\hat{a} = 1.1228$ $\hat{b} = 50.000$	126.9055	261.8111	253.9238	265.5370	259.6740	0.1580	0.8812	0.1251
OEHL-Ga	$\hat{a} = 2.7772$ $\hat{\lambda} = 3.8392$ $\hat{a} = 0.4537$ $\hat{b} = 20.000$	122.0128	252.0256	244.1383	255.7515	249.8885	0.1035	0.6276	0.1029
OEHL-N	$\hat{a} = 20.000$ $\hat{\lambda} = 20.000$ $\hat{\mu} = 10.5578$ $\hat{\sigma} = 9.0186$	135.8341	279.6683	271.7810	283.3942	277.5312	0.417	2.5334	0.1678
EE	$\hat{\gamma} = 1.7095$ $\hat{\beta} = 0.7028$	122.2436	251.9872	244.5420	255.6250	247.4187	0.1123	0.6709	0.1094
Kum-W	$\hat{a} = 1.5116$ $\hat{b} = 1.4450$ $\hat{c} = 1.0493$ $\hat{\lambda} = 0.4801$	122.1244	252.2489	244.3615	255.9747	250.1118	0.1079	0.6484	0.0981
TE-W	$\hat{\lambda} = 11.7088$ $\hat{k} = 1.4695$ $\hat{\gamma} = 14.7840$ $\hat{\theta} = 0.6167$	123.4216	252.8432	246.9266	255.6376	251.2404	0.3811	1.8959	0.1635
HLEIW	$\hat{\alpha} = 3.3326$ $\hat{\theta} = 0.3540$ $\hat{\rho} = 20.000$	126.2039	258.4077	252.4911	261.2022	256.8050	0.2256	1.315	0.1018
NE-E	$\hat{\lambda} = 0.7144$	127.1143	256.2287	254.2557	257.1601	255.6944	0.5707	2.9876	0.1663
W	$\hat{\beta} = 0.4412$	211.0708	426.1416	422.1964	428.0045	425.0730	8.6855	39.039	0.5921

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$\hat{\eta} = 29.1262$

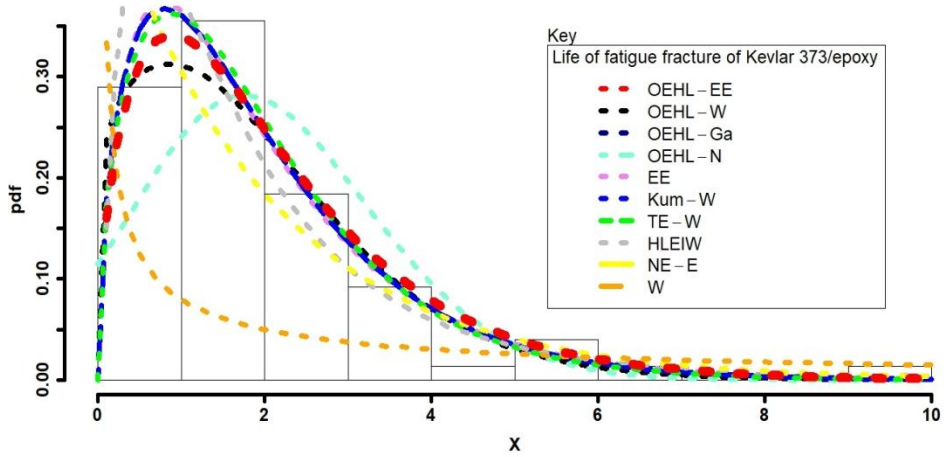


Figure 11: Plot of pdf for the estimates of the fitted models to dataset II (life of fatigue fracture of Kevlar 373/epoxy)

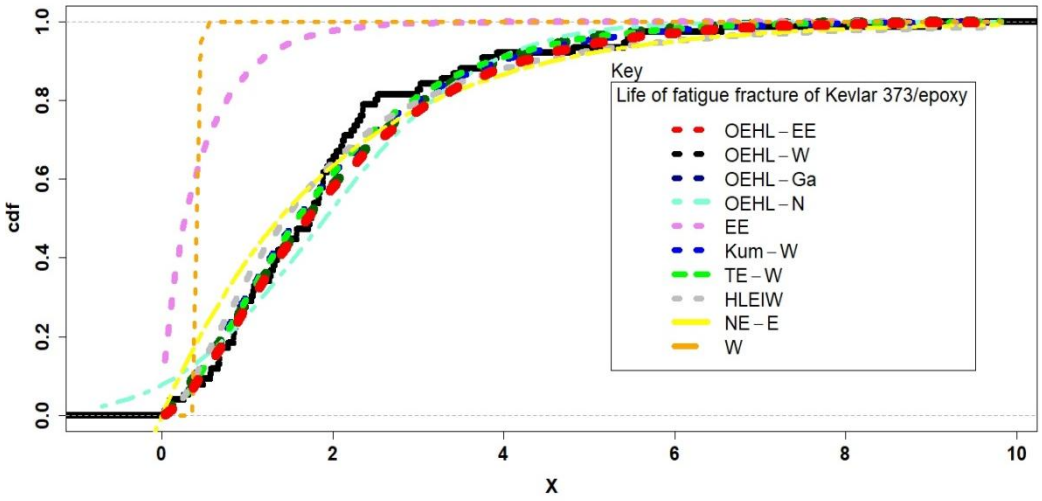


Figure 12: Plot of cdf of the fitted models and empirical cdf to dataset II (life of fatigue fracture of Kevlar 373/epoxy)

Discussion of the Real-life Data Results

Table 8 and 9 show the estimates obtained using the maximum likelihood (ML) estimation method and the goodness-of-fits obtained from different model selection criteria which include –LL, AIC, CAIC, HQIC, BIC,W^*, A^* and D^*. These results in Table 8 and 9 are obtained when the proposed OEHL-EE model and some of existing competing models are fitted to the two considered datasets (real-life datasets I and II). The results in Table 8 and 9 show that the proposed OEHL-EE distribution has the smallest values of the goodness-of-fit statistics, –LL, AIC, CAIC, HQIC, BIC,W^*, A^* and D^*, when compared with other competing models which include OEHL-EE, OEHL-EW, OEHL-Ga, OEHL-N, EE, Kum-W, TE-W, HLEIW, NE-E, W, G, and IPCS. This implies that the proposed OEHL-EE distribution outperformed other competing distributions and could be chosen as the best model compared to the other models used for fitting the two datasets. These results show that the extension of some of the existing models to develop a new model that best fit a given dataset is very useful in increasing the skewness and flexibility of the model as shown in the research done by Gupta

and Kundu (2001), Afify et al., (2017), Ishaq et al. (2023), Eze and Yahya (2025a; 2025b) etc Moreover, the formation of a new distribution from the existing one is mostly done by introducing additional shape parameters to the baseline model. Furthermore, since these two datasets are positively skewed, the results in Table 8 and 9 are evidence that the OEHL-EE distribution is an appropriate model for analyzing positively skewed data; and therefore, considered as the best among the competing distributions.

VII. Conclusion

The findings from the simulation study show that each parameter of the OEHL-EE distribution provides a specific functional role in order to determine the flexibility and adaptability of the proposed OEHL-EE distribution in fitting to data from various research areas. The shape parameter α, mainly controls the peak and heavy-tails of the distribution. It was noticed that both the median and mean increase as the α increases while there is a gradual decrease in the variance as α increases. This α behaviour provides the concentration of probability mass around the center of the density of the distribution. The high values of α provide more symmetric distribution, low coefficients of variation, and gradual increase of coefficients of kurtosis; and these indicate that there is a move toward a more peaked (leptokurtic) density of the distribution. The shape parameter λ, has influence in both the mean and median of the OEHL-EE distribution. It has been noticed that as λ increases, both the mean and median values decrease and this behaviour shows that the distribution shifts towards left and becomes denser. It has also been noticed that as λ increases, the variance decreases and this shows that the large values of λ provide narrower distributions. Moreover, the coefficients of skewness and kurtosis increase as the λ increases and this indicates a gradual shift toward right skewness and moderate tail thickness as the distribution becomes more compact. The shape parameter γ, controls the sharpness and symmetry of the distribution. It is noticed that as γ increases, both the mean and median increase while the variance and coefficient of variation decrease. This indicates that higher γ values stretch the distribution to the right. Moreover, the coefficients of skewness and kurtosis exhibit a consistent decrease as γ increases and this shows that the distribution becomes more symmetric and less heavy-tailed. Therefore, γ shapes the entire structure and smoothness of the curve of density of the distribution. The scale parameter β, only affect the magnitude of the distribution. It stretches the distribution horizontally without significantly affecting its shape. It can be seen that as β increases, the mean, median, and variance decrease and this implies a strong reduction of the distribution toward the origin. However, the coefficients of skewness and kurtosis remain nearly constant across all β values which indicates that the distribution's symmetry and peakedness are not affected by changes in scale. Hence, β serves mainly as a scale controller, adjusting the rate of reduction while keeping the shape of distribution.

Parameter estimation was accomplished using Maximum Likelihood Estimation (MLE) method. The findings show that the MLE method is good in estimating the parameters of the proposed OEHL-EE model. To show the usefulness of the OEHL-EE distribution, it was applied to two real-world datasets. The findings indicated that the OEHL-EE distribution outperformed other competing distributions such as OEHL-W, OEHL-Ga, OEHL-N, EE, Kum-W, TE-W, HLEIW, NE-E, W, G, and IPCS models on fitting to the datasets which are positively skewed. Moreover, analysis of the second dataset reaffirmed the flexibility and appropriateness of the proposed OEHL-EE distribution for modelling positively skewed datasets

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